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A GENERALIZATION OF THE CURVATURE INVARIANT

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Introduction

Let M be a manifold, and D_θ a linear connection on $T(M)$. Classically it has been shown that all local parallel vector fields, i.e., all vector fields X with $D_\theta X = 0$, must satisfy $X \in \bigcap_l \ker \nabla^l R$ where the Rimannian curvature tensor R and its covariant derivatives $\nabla^l R$ are regarded as linear maps $\nabla^l R : T(M) \to \bigotimes_{l+2} T^*(M) \otimes T(M)$. See, e.g., [1].

A related problem is the following: when is a tensor of type [1, 1] on a Riemannian or affinely connected manifold the covariant derivative of a vector field?

If E is a vector bundle and D_θ a connection on E, then the analogous questions can be asked. We study both questions in this general setting. In this context the maps $\nabla^l R$ no longer exist. However, we introduce a set of invariants $\Theta^{(l)}$ which are called higher order curvatures and serve the same purpose in this context. The definition is completely general; we associate to any differential operator $\underline{E} \xrightarrow{D} \underline{F}$ a sequence of $\mathcal{O}$-linear maps $\Theta^{(l)}(D) : E \to G_{l,0}$ where $G_{l,0}$ is canonically defined. (E and F are vector bundles.) In the present context $\Theta^{(1)}(D_\theta) = \Theta$ is the classical curvature. It is not true, however, that $\Theta^{(l)} = \nabla^{l-1}\Theta$ when $E = T(M)$, but they do have a close relationship as we shall see. Moreover, in the appropriate sense the $\Theta^{(l)}(D_\theta)$ are covariant derivatives of Θ and obey Bianchi-type identities.

The $\Theta^{(l)}$ also play a role in the study of the nonhomogeneous equation $D_\theta f = \alpha$. In fact when D_θ has constant rank, it is shown that if $E' = \bigcap_l \ker \Theta^{(l)}$, D_θ restricts to a flat connection on E'. This allows us to reduce the study of $H(M, \omega)$, where ω is the sheaf of germs of local solutions of $D_\theta f = 0$, to the case where D_θ is flat. Our preliminary calculation of $H(M, \omega)$ yields satisfactory results in two cases.

a) When the base manifold M of E is simply connected.

b) When M is a Riemannian manifold of strictly positive or strictly negative sectional curvature, and D_θ is the Riemannian connection on $T(M)$.

Some of the results of this paper were announced in [3].

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0. Preliminaries

We shall make use of several notions arising in the theory of overdetermined systems without defining them in the text. For a full account of the ideas involved one should consult [2a], [6a], or [7]. For easy reference we recall them here.

Let E be a vector bundle. (We work entirely in the C^∞ category.) Then we write $J_k(E)$ for the bundle of k-jets. The fibre $J_k(E)$ over a point x is the quotient of the sheaf of germs of sections of E by the subset of germs of sections vanishing to order $k+1$ at x. We note that J_k is a functor from the category of C^∞ vector bundles and bundle morphisms into itself. We write j_k for the differential operator

$$j_k : \underline{E} \to J_k(E)$$

which takes a germ of a section of E to its quotient.

The jet bundles enjoy a universal property with respect to differential operators. Namely, let $D: \underline{E} \to \underline{F}$ be a differential operator of order k. Then there is a unique bundle map $\rho(D): J_k(E) \to F$ such that $D = \rho(D) \circ j_k$.

Note that we have a natural injection $0 \to S^k T^* \otimes E \xrightarrow{\varepsilon} J_k(E)$ such that

$$0 \longrightarrow S^k T^* \otimes E \xrightarrow{\varepsilon} J_k(E) \xrightarrow{\pi_{l-1}} J_{k-1}(E) \longrightarrow 0$$

is exact. We define the symbol of an operator D with bundle map $\rho(D)$ to be the composition $\sigma(D) = \rho(D) \circ \varepsilon$.

Given an operator $D: \underline{E} \to \underline{F}$ of order k, the composition $j_l \circ D : \underline{E} \to J_l(F)$ is of order $k+l$. It has a corresponding bundle map $\rho_l(D)$ called the lth prolongation of D. It is the unique map such that

$$\begin{array}{ccc} J_{k+l}(E) & \xrightarrow{\rho_l(D)} & J_l(F) \\ {\scriptstyle j_{k+l}}\big\downarrow & & \big\uparrow{\scriptstyle j_l} \\ \underline{E} & \xrightarrow{D} & \underline{F} \end{array}$$

commutes. Similarly the prolonged symbol is the composition $\sigma_l(D) = \rho_l(D) \circ \varepsilon$ which goes into the sub-bundle $S^l T^* \otimes F$ of $J_l(F)$, i.e.,

$$\sigma_l(D) : S^{k+l} T^* \otimes E \to S^l T^* \otimes F \ .$$

Of particular interest are the kernels of these maps. Thus we define

$$R_k = \ker \rho(D) \ , \qquad R_{k+l} = \ker \rho_l(D) \ ,$$

which are called the equation and prolonged equations, respectively. Further

$$g_k = \ker \sigma(D) , \qquad g_{k+l} = \ker \sigma_l(D)$$

are called the symbol and prolonged symbol, respectively.

Since $R_{k+l} \subset J_{k+l}(E)$, we can restrict π_{k+l-1} to R_{k+l} and have $\pi_{k+l-1}R_{k+l} \subset R_{k+l-1}$. Hence we can state the important

Definition. An operator D is formally integrable if R_{k+l} is a vector bundle for every $l \geq 0$ and $\pi_{k+l-1}: R_{k+l} \to R_{k+l-1}$ is surjective for every $l \geq 1$.

We also have

Definition. An operator D is regular if R_{k+l} is a vector bundle for every $l \geq 0$.

We will need the operator

$$D: \underline{\bigwedge^r T^* \otimes J_k(E)} \to \underline{\bigwedge^{r+1} T^* \otimes J_{k-1}(E)}$$

which is characterized by the following conditions:

(1) The sequence

$$0 \longrightarrow \underline{E} \longrightarrow \underline{J_k(E)} \xrightarrow{D} \underline{T^* \otimes J_{k-1}(E)}$$

is exact.

(2) If s is a section of $\bigwedge^p T^* \otimes J_k(E)$, and α is a differential form of degree q, then $D(\alpha \wedge s) = d\alpha \wedge \pi_{k-1}s + (-1)^q \alpha \wedge Ds$.

We have $D^2 = 0$, $D(\underline{\bigwedge^r T^* \otimes R_{k+l+1}}) \subset \underline{\bigwedge^{r+1} T^* \otimes R_{k+l}}$, and if $\delta: \bigwedge^r T^* \otimes S^k T^* \otimes E \to \bigwedge^{r+1} T^* \otimes S^{k-1} T^* \otimes E$ is the restriction of $-D$, then

$$\delta(\bigwedge^r T^* \otimes g_{k+l+1}) \subset \bigwedge^{r+1} T^* \otimes g_{k+l} .$$

This leads to the complexes

$$\begin{aligned} g_m \xrightarrow{\delta} T^* \otimes g_{m-1} \xrightarrow{\delta} \bigwedge^2 T^* \otimes g_{m-2} \cdots \xrightarrow{\delta} \bigwedge^r T^* \otimes g_{m-r} \longrightarrow \\ \cdots \xrightarrow{\delta} \bigwedge^{m-k} T^* \otimes g_k \longrightarrow \bigwedge^{m-k+1} T^* \otimes S^{k-1} T^* \otimes E \end{aligned}$$

where $m \geq k$. g_k is said to be involutive if these sequences are exact.

Next we construct the second Spencer complex. Let $C^r_m = (\bigwedge^r T^* \otimes R_{m+1}) / \delta(\bigwedge^{r-1} T^* \otimes g_{m+2})$. Then the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\bigwedge^r T^* \otimes g_{m+2}} & \longrightarrow & \underline{\bigwedge^r T^* \otimes R_{m+2}} & \xrightarrow{\pi_{m+1}} & \underline{\bigwedge^r T^* \otimes R_{m+1}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle -\delta} & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle \tilde{D}} & & \\ 0 & \longrightarrow & \underline{\delta(\bigwedge^r T^* \otimes g_{m+2})} & \longrightarrow & \underline{\bigwedge^{r+1} T^* \otimes R_{m+1}} & \longrightarrow & \underline{C^{r+1}_m} & \longrightarrow & 0 \end{array}$$

induces an operator $\tilde{D}$ which factors through $\underline{C^r_m}$ and therefore defines an operator

$$\hat{D}: \underline{C^r_m} \to \underline{C^{r+1}_m}\,.$$

Thus we obtain the Spencer complex

$$0 \longrightarrow \Theta \longrightarrow \underline{C^0_m} \xrightarrow{\hat{D}} \underline{C^1_m} \xrightarrow{\hat{D}} C^2_m \longrightarrow \cdots$$

Suppose R_m is formally integrable and $g_m = 0$. It follows that $g_{m+l} = 0$, $l \geq 0$. Then $\pi^{-1}_{m+1}: \wedge^r T^* \otimes R_{m+1} \to \wedge^r T^* \otimes R_{m+2}$ is an isomorphism, and the Spencer complex is given by

$$0 \longrightarrow \Theta \longrightarrow R_{m+1} \xrightarrow{\hat{D}} T^* \otimes R_{m+1} \xrightarrow{\hat{D}} \wedge^2 T^* \otimes R_{m+1} \longrightarrow \cdots,$$

where $\hat{D} = D \circ \pi^{-1}_{m+1}$. Thus $\hat{D}^2 = 0$, and if $fs \in R_{m+1}$ where f if a function, then we have $\hat{D}(fs) = D(f\pi^{-1}_{m+1}s) = df \wedge s + fD\pi^{-1}_{m+1}s = df \wedge s + f\hat{D}s$. It follows that $\sigma(\hat{D}) = \mathrm{id}$, i.e., that $\hat{D}$ is a connection on R_{m+1}. Similarly if $\alpha \in \wedge^q T^*$, then $\hat{D}(\alpha \wedge s) = D(\alpha \wedge \pi^{-1}_{m+1}s) = d\alpha \wedge s + (-1)^q \alpha \wedge \hat{D}s$. It follows that $\hat{D}^2 = 0$ is the curvature of $\hat{D}$. Therefore one may choose flat frames $s_1, \cdots, s_n$ for R_{m+1}. Now however, since $\hat{D}s_i = 0$, we have $D \circ \pi^{-1}_{m+1}s_i = 0$. It follows that $\pi^{-1}_{m+1}s_i = j_{m+2}(e_i)$ where e_i are sections in E and hence $s_i = \pi_{m+1} \circ \pi^{-1}_{m+1} \circ s_i = j_{m+1}(e_i)$. The foregoing is a special case of a more general result due to D. C. Spencer. For details see [6a].

Finally, we recall [2a]

Theorem. *Let $\varphi: J_k(F) \to F$ be a linear map with $R_k = \ker \varphi$. Then it is formally integrable if*

i) *R_{k+1} is a vector bundle,*
ii) *$\pi_k: R_{k+1} \to R_k$ is surjective,*
iii) *g_k is 2-acyclic.*

1. Higher order curvature

1.1. *Definitions and basic properties.* Let E and F be vector bundles and D a differential operator of order k $D: \underline{E} \to \underline{F}$. We define $D^{(l,i)}$, $l \geq 0$, $0 \leq i \leq l + k - 1$ by requiring that the following diagram commute:

$$\begin{tikzcd} J_{k+l}(E) \arrow[r, "\rho_l(D)"] & J_l(F) \arrow[dr, "\tilde{\omega}_i"] & \\ E \arrow[u, "j_{k+l}"] \arrow[r, "D"] & F \arrow[u, "j_l"] \arrow[r, "D^{(l,i)}"] & J_l(F)/\rho_l(D)(J^i_{k+l}(E)) \end{tikzcd}$$

where $J^i_{k+l}(E)$ is the kernel of $\pi_i: J_{k+l} \xrightarrow{\pi_i} J_i(E)$, and $\tilde{\omega}_i$ is the natural projection. We define $\Theta^{(l,i)}(D) = \rho(D^{(l,i)} \circ D)$, and set $\Theta^{(l,i)} = \Theta^{(l,i)}(D)$ when D is understood. When $i = 0$, we write $\Theta^{(l)} = \Theta^{(l,0)}$ and call $\Theta^{(l)}$ the lth curvature of D.

Proposition 1.1. *The operator $D^{(l,i)} \circ D$ is a differential operator of order i.*

Proof. By construction, $\rho(D^{(l,i)} \circ D) = \tilde{\omega}_i \circ \rho_l(D)$ which vanishes on $J^i_{k+l}(E)$, establishing the proposition.

In fact the diagram

$$\begin{tikzcd}
0 \arrow[r, dash] & J^i_{k+l}(E) \arrow[r] & J_{k+l}(E) \arrow[r, "\pi_i"] \arrow[dr, "\tilde{\omega}_i \circ \rho_l(D)"'] & J_i(E) \arrow[r] \arrow[d, dashed, "\Omega^{(i,l)}(D)"] & 0 \\
& & & J_l(F)/\rho_l(D)(J^i_{k+l}(E)) &
\end{tikzcd}$$

induces a map $\Omega^{(l,i)}(D)$, and we observe

$$\Theta^{(l,i)}(D) = \rho(D^{(l,i)} \circ D) = \tilde{\omega}_i \circ \rho_i(D) = \Omega^{(l,i)}(D) \circ \pi_i \,.$$

Henceforth we will identify $\Theta^{(l,i)}(D)$ and $\Omega^{(l,i)}(D)$ and, where convenient, define $\Theta^{(l,i)}(D) : J_r(E) \to J_l(F)/\rho_l(D)(J^i_{k+l}(D))$ where $r > i$ by $\Theta^{(l,i)} \circ \pi_i$. In particular $\Theta^{(l)} = D^{(l,0)} \circ D$, and we have

Corollary. *$\Theta^{(l)}$ is a zero order differential operator between families of vector spaces.*

Proposition 1.2. *If $0 \leq m < l$ and $-1 \leq i < k + m$, then the diagram*

(1.1)

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & R^{k+m}_{k+l} \arrow[r] \arrow[d, "i"] & J^{k+m}_{k+l}(E) \arrow[r, "\rho_l(D)"] \arrow[d, "i"] & J^m_l(F) \arrow[r, "\tilde{\omega}_{k+m}"] \arrow[d, "i"] & J^m_l(F)/\rho_l(D)(J^{k+m}_{k+l}(D)) \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & R^i_{k+l} \arrow[r] \arrow[d, "\pi_{k+m}"] & J^i_{k+l}(E) \arrow[r, "\rho_l(D)"] \arrow[d, "\pi_{k+m}"] & J_l(F) \arrow[r, "\tilde{\omega}_i"] \arrow[d, "\pi_m"] & J_l(F)/\rho_l(D)(J^i_{k+l}(D)) \arrow[r] \arrow[d, "\pi_m"] & 0 \\
0 \arrow[r, dash] & R^i_{k+m} \arrow[r, dash] \arrow[d] & J^i_{k+m}(E) \arrow[r, "\rho_m(D)"] \arrow[d] & J_m(F) \arrow[r, "\tilde{\omega}_i"] \arrow[d] & J_m(F)/\rho_m(D)(J^i_{k+m}(D)) \arrow[r] \arrow[d] & 0 \\
& h^i_{k+m,l} \arrow[d] & 0 & 0 & 0 & \\
& 0 & & & &
\end{tikzcd}$$

is exact where the top row and right and left columns are induced, and we have adopted the convention $J^{-1}_k(E) = J_k(E)$.

Proof. Clear by a diagram chase.

The main interest is the case $m = l - 1$, for it gives us specific information on the structure of $J_l(F)/\rho_l(J^i_{k+l}(E))$. Namely we have

Proposition 1.3. *The sequence*

$$0 \longrightarrow h^i_{k+m,l} \longrightarrow \frac{S^lT^* \otimes F}{\sigma_l(D)(S^{l+k}T^* \otimes E)} \longrightarrow \frac{J_l(F)}{\rho_l(D)(J^i_{k+l}(E))} \longrightarrow \frac{J_{l-1}(F)}{\rho_{l-1}(D)(J^i_{k+l-1}(E))} \longrightarrow 0$$

is exact. When $\pi^i_{k+m}\colon R^i_{k+l} \to R^i_{k+m}$ *is surjective,*

$$0 \longrightarrow \frac{S^lT^* \otimes F}{\sigma_l(D)(S^{l+k}T^* \otimes F)} \longrightarrow \frac{J_l(F)}{\rho_l(D)(J^i_{k+l}(E))} \longrightarrow \frac{J_{l-1}(F)}{\rho_{l-1}(D)(J^i_{k+l-1}(E))} \longrightarrow 0$$

is exact.

Proof. If suffices to observe that $\rho_l(D)_{|J^{k+l-1}_{k+l}(E)} = \sigma_l(D)$ under the identification $J^{k+l-1}_{k+l}(E) = S^lT^* \otimes E$. The rest is a diagram chase of (1.1).

Proposition 1.4. *If $i - k < m < l$, we have $\pi_m \circ \Theta^{(l,i)} = \Theta^{(m,i)}$.*

Proof. We have $\pi_m \circ \Theta^{(l,i)} = \pi_m \circ \tilde{\omega}_i \circ \rho_l(D) = \tilde{\omega}_i \circ \pi_m \circ \rho_l(D) = \tilde{\omega}_i \circ \rho_m(D) \circ \pi_{k+m} = \Theta^{(m,i)} \circ \pi_{k+m} = \Theta^{(m,i)}$.

Proposition 1.5. *We have an exact sequence*

$$0 \longrightarrow \pi_i(R^j_{k+l}) \longrightarrow J^j_i(E) \xrightarrow{\Theta^{(l,i)}} J_l(F)/\rho_l(D)(J^i_{k+l}(E)) \longrightarrow J_l(F)/\rho_l(D)(J^j_{k+l}(E)) \longrightarrow 0\,.$$

Proof. Follows by a diagram chase from the diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & R^i_{k+l} & \longrightarrow & J^i_{k+l}(E) & \xrightarrow{\rho_l(D)} & J_l(F) & \xrightarrow{\omega_i} & J_l(F)/\rho_l(D)(J^i_{k+l}(E)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & R^j_{k+l} & \longrightarrow & J^j_{k+l}(E) & \xrightarrow{\rho_l(D)} & J_l(F) & \xrightarrow{\tilde{\omega}_i} & J_l(F)/\rho_l(D)(J^j_{k+l}(D)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi_i} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^j_i & \longrightarrow & J^j_i(E) & & 0 & & 0 & & \\
 & & & & \downarrow & & & & & & \\
 & & & & 0 & & & & & &
\end{array}$$

Corollary. *If $\Theta^{(l)} = 0$, then $J_l(F)/\rho_l(D)(J^0_{k+l}(E))$ and $J_l(F)/\rho_l(D)(J_{k+l}(E))$ are canonically isomorphic.*

Proof. Follows from Proposition 1.5 with $i = 0$, $j = -1$.

Theorem 1.1. *Let R_k be the equation of D, and R_{k+l} its prolonged equations. Then $\pi_m(R_{k+l}) = \ker \Theta^{(l,m)}(D)$.*

Proof. Follows from Proposition 1.5 with $j = -1$.

1.2. *Geometric meaning of higher order curvature.* a) Bianchi identities. We expect $\Theta^{(l)}(D)$ to have geometric meaning whenever D does. In the remainder of this chapter we explore in some detail the meaning of $\Theta^{(l)}(D_\theta)$ where D_θ is a covariant differential operator. Recall that D_θ can be thought of in the following way: Let E be a differential operator, and θ a splitting of the sequence

$$0 \longrightarrow T^* \otimes E \xrightarrow{i} J_1(E) \xrightarrow{\pi_0} E \longrightarrow 0 \,.$$
$$\underset{\rho(D_\theta)}{\longleftarrow} \qquad \underset{\theta}{\longleftarrow}$$

θ induces a map $\rho(D_\theta): J_1(E) \to T^* \otimes E$ with $\rho(D_\theta) \circ i = \text{id}$. $\rho(D_\theta)$, in turn, defines a first order operator $D_\theta: \underline{E} \to \underline{T^* \otimes E}$ whose symbol $\sigma(D_\theta): T^* \otimes E \to T^* \otimes E$ is the identity.

Our first point is the following:

Theorem 1.2. *Let Θ be the classical curvature regarded as a section of* $\operatorname{Hom}(E, \wedge^2 T^* \otimes E)$. *Then $\Theta = \Theta^{(1)}(D_\theta)$ up to a canonical isomorphism. Further, up to the same isomorphism*

$$D_\theta^{(1,0)} = D_\theta : T^* \otimes E \longrightarrow \wedge^2 T^* \otimes E \,.$$

We need

Lemma 1.1. *The sequence*

$$J_{l+1}^1(E) \xrightarrow{\rho_l(D_\theta)} J_l^0(T^* \otimes E) \xrightarrow{\tilde{\omega}_0} J_l(T^* \otimes E)/\rho_l(D_\theta)(J_{l+1}^0(E)) \longrightarrow 0$$

is exact

Proof. We first must show that $\rho_l(D_\theta): J_{l+1}^1(E) \to J_l^0(T^* \otimes E)$ is defined, i.e., that $\rho_l(D_\theta)(J_{l+1}^1(E)) \subset J_l^0(T^* \otimes E)$. Let $\alpha \in J_{l+1}^1(E)(x_0)$ and choose $f \in \underline{E}_{x_0}$ such that $j_{l+1}f = \alpha$. Then, since D_θ is first order, $(D_\theta f)(x_0) = 0$. We have $[\rho_l(D_\theta)\alpha](x_0) = [j_l \circ D_\theta f](x_0)$ and hence $[\pi_0(\rho_l(D_\theta)\alpha)](x_0) = [\pi_0(j_l \circ D_\theta f)](x_0) = (D_\theta f)(x_0) = 0$ which is to say $\rho_l(D_\theta)\alpha \in J_l^0(T^* \otimes E)$.

It is clear by definition of the maps involved that $\tilde{\omega}_0 \circ \rho_l(D_\theta) = 0$. We now show that $\tilde{\omega}_0$ is surjective. Namely let $\beta \in [J_l(T^* \otimes E)/\rho_l(D_\theta)(J_{l+1}^0(E))]_{x_0}$ and let $\alpha \in J_l(T^* \otimes E)_{x_0}$ be a representative of β. Since the symbol of D_θ is an isomorphism, we can find $f \in \underline{E}_{x_0}$ with $f(x_0) = 0$ and $(D_\theta f)(x_0) = \pi_0\alpha$. Since $j_{l+1}f \in J_{l-1}^0(E)_{x_0}$, we have $\tilde{\omega}_0(j_l \circ D_\theta f)(x_0) = (\tilde{\omega}_0 \circ \rho_l(D_\theta) \circ j_{l+1}f)(x_0) = 0$ and hence $\tilde{\omega}_0(\alpha - j_l(D_\theta f)(x_0)) = \tilde{\omega}_0(\alpha) = \beta$. Since $\alpha - j_l(D_\theta f)(x_0) \in J_l^0(T^* \otimes E)_{x_0}$, this establishes surjectivity of $\tilde{\omega}_0$.

It remains to establish exactness at $J_l^0(T^* \otimes E)$. Suppose $\beta \in J_l^0(T^* \otimes E)_{x_0}$ and $\beta \in \ker \tilde{\omega}_0$. This means that $\beta = \rho_l(D_\theta)\alpha$ for some $\alpha \in J_{l+1}^0(E)_{x_0}$. Let

$j_{l+1}f = \alpha$ where $f \in \underline{E}_{x_0}$, and note that $f(x_0) = 0$. Further, $(D_\theta f)_{(x_0)} = \pi_{0_l}\beta = 0$. But observe that $[\sigma(D_\theta) \circ j_1 f](x_0) = [\rho(D_\theta) \circ j_1 f](x_0) = (D_\theta f)(x_0) = 0$ so, since $\sigma(D_\theta)$ is injective, $(j_1 f)(x_0) = 0$. It follows that $\alpha \in J^1_{l+1}(E)$.

Corollary. *The sequence*

$$S^2T^* \otimes E \xrightarrow{\delta} T^* \otimes T^* \otimes E \xrightarrow{\tilde{\omega}_0} J_1(T^* \otimes E)/\rho_1(D_\theta)(J^0_2(E)) \longrightarrow 0$$

is exact, inducing an isomorphism $h\colon \bigwedge^2 T^* \otimes E \to J_1(T^* \otimes E)/\rho_1(D_\theta)(J^0_2(E))$. *If* δ *is the projection* $\delta\colon T^* \otimes T^* \otimes E \to \bigwedge^2 T^* \otimes E$, *and* $\alpha \in [T^* \otimes T^* \otimes E]_x$ *is a representative of* $\beta = \delta(\alpha)$, *then* $h(\beta) = \tilde{\omega}_0(\alpha)$.

Proof of Theorem 1.2. It is enough to show that $D^{(1,0)}_\theta = h \circ D_\theta$ where $D_\theta\colon \underline{T^* \otimes E} \to \bigwedge^2 T^* \otimes E$, since it immediately follows that $\Theta^{(1)}(D_\Theta) = D^{(1,0)}_\theta \circ D_\theta = h \circ D_\theta \circ D_\theta = h \circ \Theta$. Recall that $D_\theta\colon \underline{T^* \otimes E} \to \bigwedge^2 \underline{T^* \otimes E}$ may be defined in the following manner. Namely, if $\alpha \otimes e \in T^* \otimes E_{x_0}$, then $D_\theta(\alpha \otimes e) = d\alpha \otimes e - \alpha \wedge D_\theta e$. Accordingly, let $\alpha \otimes e \in \underline{T^* \otimes E}_{x_0}$. We choose e with $e(x_0) \neq 0$. Since the symbol of D_θ is the identity, we can choose $f \cdot e \in \underline{E}_{x_0}$ such that $f(x_0) = 0$ and $D_\theta(f \cdot e)(x_0) = (\alpha \otimes e)(x_0)$. Since $f(x_0) = 0$, we have $D^{(1,0)}_\theta \circ D_\theta(f \cdot e)(x_0) = \Theta^{(1)}(f \cdot e)(x_0) = 0$. Hence $\tilde{\omega}_0(j_1(\alpha \otimes e - D_\theta(f \cdot e)))(x_0) = \tilde{\omega}_0 \circ j_1(\alpha \otimes e)(x_0)$. We have $j_1(\alpha \otimes e - D_\theta(f \cdot e))(x_0) = j_1(\alpha \otimes e - df \otimes e)(x_0) - j_1(fD_\theta e)(x_0) = j_1((\alpha - df) \otimes e)(x_0) - j_1(fD_\theta e)(x_0)$. Since $j_1(fD_\theta e)(x_0)$ and $j_1((\alpha \otimes e - D_\theta(f \cdot e))(x_0)$ lie in $T^* \otimes T^* \otimes E$, $j_1((\alpha - df) \otimes e)(x_0) \in T^* \otimes T^* \otimes E$. It follows that $(\alpha - df)(x_0) = 0$.

Notice now that for any $\beta \otimes e \in T^* \otimes E$ with $\beta(x_0) = 0$, we have $\delta \circ j_1(\beta \otimes e)(x_0) = d\beta \otimes e(x_0)$. It follows that $\delta(j_1(\alpha \otimes e) - D_\theta(f \cdot e))(x_0) = \delta \circ j_1((\alpha - df) \otimes e)(x_0) - \delta \circ j_1(fD_\theta e)(x_0) = (d\alpha \otimes e)(x_0) - df \wedge D_\theta e(x_0) = d\alpha \otimes e(x_0) - \alpha \wedge D_\theta e(x_0)$ establishing the theorem.

D_θ is extended to $\underline{\wedge\, T^* \otimes E}$ in the following manner. Let $\alpha \otimes e \in \bigwedge^i T^* \otimes E$. Then $D_\theta(\alpha \otimes e) = d\alpha \otimes e + (-1)^i \alpha \wedge D_\theta e \in \underline{\bigwedge^{i+1} T^* \otimes E}$. Notice that $D^2_\theta(\alpha \otimes e) = D_\theta[d\alpha \otimes e + (-1)^i \alpha \wedge D_\theta e] = d^2\alpha \otimes e + (-1)^{i+1} d\alpha \wedge D_\theta e + (-1)^i d\alpha \wedge D_\theta e + (-1)^{2i} \alpha \wedge D^2_\theta e = \alpha \wedge \Theta e$. We will later need the following generalization of Theorem 1.2:

Theorem 1.3. *Consider the differential operator* $D^{(1,0)}_\theta = \varpi_0 \circ j_1\colon \underline{\bigwedge^i T^* \otimes E} \to J_1(\bigwedge^i T^* \otimes E)/\rho_1(D_\theta)(J^0_2(\bigwedge^{i-1} T^* \otimes E))$. *Call the second bundle* $G^{(i)}_{1,0}$. *Then there is a canonical isomorphism* $h^{(i)}\colon \bigwedge^{i+1} T^* \otimes E \to G^{(i)}_{1,0}$ *and* $D^{(1,0)}_\theta = h^{(i)} \circ D_\theta$.

Proof. Consider the commutative diagram where $i \geq 2$.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
J_3^2(\wedge^{i-2}T^* \otimes E) \arrow[r, "\delta"] \arrow[d] & J_2^1(\wedge^{i-1}T^* \otimes E) \arrow[r, "\delta"] \arrow[d] & J_1^0(\wedge^{i}T^* \otimes E) \arrow[r, "\delta"] \arrow[d] & \wedge^{i+1}T^* \otimes E \arrow[r] \arrow[d, "h^{(i)}"] & 0 \\
J_3^1(\wedge^{i-2}T^* \otimes E) \arrow[r, "\rho_2(D_\theta)"] \arrow[d, "\pi_2"] & J_2^0(\wedge^{i-1}T^* \otimes E) \arrow[r, "\rho_1(D_\theta)"] \arrow[d, "\pi_1"] & J_1(\wedge^{i}T^* \otimes E) \arrow[r, "\tilde{\omega}_0"] \arrow[d, "\pi_0"] & G_{1,0}^{(i)} \arrow[r] \arrow[d] & 0 \\
J_2^1(\wedge^{i-2}T^* \otimes E) \arrow[r, "\delta"] \arrow[d] & J_1^0(\wedge^{i-1}T^* \otimes E) \arrow[r, "\delta"] \arrow[d] & \wedge^{i}T^* \otimes E \arrow[r] \arrow[d] & 0 & \\
0 & 0 & 0 & &
\end{tikzcd}$$

The top and bottom rows are exact, and all columns except possibly the last, which is induced by the diagram, are exact. The map $\rho_2(D_\theta)$ restricts to $J_3^1(\wedge^{i-2}T^* \otimes E)$ and the image of the restriction lies in $J_2^0(\wedge^{i-1}T^* \otimes E)$ because D_θ is a first-order operator. Since $\rho_1(D_\theta) \circ \rho_2(D_\theta) = \rho_1(D_\theta^2) = \rho_1(\Theta)$ is a first-order operator, it must vanish on $J_3^1(\wedge^{i-2}T^* \otimes E)$. Since exactness in the rest of the middle row is clear, the middle row is a complex. Exactness at $J_2^0(\wedge^{i-1}T^* \otimes E)$ follows by a diagram chase. Note that exactness of the last column is not required in this chase. Now that everything else is exact exactness of the last column and specifically the isomorphism $\wedge^{i+1}T^* \otimes E \xrightarrow{h^{(i)}} G_{1,0}^{(i)}$ follows by a diagram chase. Explicity if $\beta \in \wedge^{i+1}T^* \otimes E$ and $\delta(\alpha) = \beta$, then $h^{(i)}\beta = \tilde{\omega}_0\alpha$.

We now proceed as in Theorem 1.2. Let $\alpha \otimes e \in \wedge^i T^* \otimes E_{x_0}$. We choose e with $e(x_0) \neq 0$. Choose $\beta \otimes e \in \wedge^{i-1}T^* \otimes E_{x_0}$ with $d\beta(x_0) = \alpha$ and $\beta(x_0) = 0$. Noticing that $D_\theta(\beta \otimes e)(x_0) = (\alpha \otimes e)(x_0)$, we have $D_\theta^{(1,0)} \circ D_\theta(\beta \otimes e)(x_0) = \Theta^{(1)}(\beta \otimes e)(x_0) = 0$. Hence $D_\theta^{(1,0)}(\alpha \otimes e - D_\theta(\beta \otimes e))(x_0) = D_\theta^{(1,0)}(\alpha \otimes e)$. We therefore replace $\alpha \otimes e$ by $\alpha \otimes e - D_\theta(\beta \otimes e)$. We have $j_1(\alpha \otimes e - D_\theta(\beta \otimes e))(x_0) = j_1(\alpha \otimes e - d\beta \otimes e)(x_0) - j_1((-1)^{i-1}\beta \wedge D_\theta e)(x_0)$. Both terms lie in $T^* \otimes \wedge^i T^* \otimes E$. Remark that for any $\gamma \otimes e \in \wedge^i T^* \otimes E$ with $\gamma(x_0) = 0$, $\delta \circ j_1(\gamma \otimes e)(x_0) = d\gamma \otimes e(x_0)$. Applying this fact we have $\delta(j_1((\alpha \otimes e) - D_\theta(\beta \otimes e)))(x_0) = \delta \circ j_1((\alpha - d\beta) \otimes e)(x_0) + (-1)^i \delta \circ j_1(\beta \wedge D_\theta e)(x_0) = (d\alpha \otimes e)(x_0) + (-1)^i(\alpha \wedge D_\theta e)(x_0)$, establishing the theorem.

Now consider the sequence

$$\underline{E} \xrightarrow{D_\theta} \underline{T^* \otimes E} \xrightarrow{D_\theta} \wedge^2 T^* \otimes E \xrightarrow{D_\theta} \wedge^3 T^* \otimes E\,.$$

Let A be a bundle map $A : E \to \wedge^2 T^* \otimes E$. Then the standard covariant derivative of A can be defined in the following way: Define A' to be the composition

$$A' \colon T^* \otimes E \xrightarrow{\mathrm{id} \otimes A} T^* \otimes \wedge^2 T^* \otimes E \xrightarrow{\wedge} \wedge^3 T^* \otimes E$$

where $\wedge$ is the obvious map. (One can check, in particular, that $\Theta' = D_\theta^2$: $T^* \otimes E \to \wedge^3 T^* \otimes E$.)

Then define

$$D_\theta A = D_\theta \circ A - A' \circ D_\theta . \tag{1.2}$$

From this point of view the Bianchi identity becomes utterly trivial:

$$D_\theta \Theta = D_\theta \circ \Theta - \Theta \circ D_\theta = D_\theta \circ D_\theta \circ D_\theta - D_\theta \circ D_\theta \circ D_\theta = 0 .$$

We wish to generalize this formula to find a way to covariantly differentiate the higher-order curvatures $\Theta^{(l)}$ by some operator $\tilde{D}_{\theta,l}$ with $\tilde{D}_{\theta,l}\Theta^{(l)} = 0$.

Clearly, we wish that the domain of $\tilde{D}_{\theta,l}$ be Hom $(E, G_{l,0})$ where $G_{l,0} = J_l(T^* \otimes E)/\rho_l(D_\theta)(J_{l+1}^0(E))$ is the range of $\Theta^{(l)}$. We wish further that $\tilde{D}_{\theta,l}(A)$ be linear over functions and that it be expressible in a form analogous to (1.2). Finally, it would be nice if $\tilde{D}_{\theta,l}\Theta^{(l)}$ vanished for a reason analogous to the reason given for $D_\theta \Theta = 0$.

We begin with

Lemma 1.2. *Let E and F be vector bundles, and $A : E \to F$ a bundle map. Regarding A as a zero-order operator, we can take its first prolongation. Then we have $\sigma_1(A) = \mathrm{id} \otimes A$ where $\sigma_1(A) : T^* \otimes E \to T^* \otimes F$.*

Proof. Trivial.

Our next proposition allows us to take a covariant derivative of anything in Hom (E, F) given only a covariant derivative on E.

Proposition 1.6. *Let E be a vector bundle with covariant derivative D_θ, and F another vector bundle with $A : E \to F$ a bundle map. Consider the diagram*

$$\begin{tikzcd} F \arrow[rr, "j_1"] & & J_1(F) \\ E \arrow[u, "A"] \arrow[r, "D_\theta"] & T^* \otimes E \arrow[r, "\mathrm{id} \otimes A"] & T^* \otimes F . \arrow[u, "i"'] \end{tikzcd}$$

Then $\check{D}_\theta A \equiv j_1 \circ A - i \circ (\mathrm{id} \otimes A) \circ D_\theta$ is a bundle map.

Proof. Since both $j_1 \circ A$ and $i \circ (\mathrm{id} \otimes A) \circ D_\theta$ are first-order operators, it suffices to prove that their symbols are identically equal. But we have

$$\sigma(j_1 \circ A) = \sigma(j_1) \circ \sigma_1(A) = \mathrm{id} \circ (\mathrm{id} \otimes A) = \mathrm{id} \otimes A$$

and, on the other hand,

$$\sigma(i \circ (\mathrm{id} \otimes A) \circ D_\theta) = (\mathrm{id} \otimes A) \circ \sigma(D_\theta) = \mathrm{id} \otimes A$$

since $\sigma(D_\theta) = \mathrm{id}$.

Remark. $\check{D}_\theta$ is thus a first-order operator

$$\check{D}_\theta : \underline{\mathrm{Hom}\,(E, F)} \longrightarrow \underline{\mathrm{Hom}\,(E, J_1(F))}\ .$$

In the classical situation F is equipped with a first-order differential operator and composition of its bundle map with $\check{D}_\theta A$ provides the usual covariant derivative of A. In accordance with this point of view, we now focus on the bundles $G_{l,0}$ and desire to take covariant derivatives of sections of Hom $(E, G_{l,0})$. We consider the following sequence

$$\underline{E} \xrightarrow{D_\theta} \underline{T^* \otimes E} \xrightarrow{D_\theta^{(l,0)}} \underline{G_{l,0}} \xrightarrow{D'_{\theta,l}} \underline{\frac{J_1(G_{l,0})}{\rho_1(D_\theta^{(l,0)})(J^0_{l+1}(T^* \otimes E))}}$$

where $D'_{\theta,l} = \tilde{\omega}_0 \circ j_1$, and we note that $\Theta^{(1)}(D_\theta^{(l,0)}) = D'_{\theta,l} \circ D_\theta^{(l,0)}$.

We make the following definition.

Definition. Let $A : E \to G_{l,0}$ be a bundle map. Then we write

$$\tilde{D}_{\theta,l}A \equiv D'_{\theta,l} \circ A - \sigma(D'_{\theta,l}) \circ (\mathrm{id} \otimes A) \circ D_\theta \equiv \rho(D'_{\theta,l}) \circ \check{D}_{\theta,l}A\ .$$

Hence $\tilde{D}_{\theta,l}$ is a first-order differential operator and

$$\tilde{D}_{\theta,l} : \underline{\mathrm{Hom}\,(E, G_{l,0})} \longrightarrow \underline{\mathrm{Hom}\,(E, G'_{l,0})}$$

where $G'_{l,0} = J_1(G_{l,0}) / \rho_1(D_\theta^{(l,0)})(J^0_{l+1}(T^* \otimes E))$.

Notice that the A' we sought is provided by $A' = \sigma(D'_{\theta,l}) \circ (\mathrm{id} \otimes A)$.

We now state

Theorem 1.4. $\tilde{D}_{\theta,l}\Theta^{(l)} = 0$.

Proof. Remembering that $\Theta^{(l)} = D^{(l,0)} \circ D_\theta$ we need only to prove $(\Theta^{(l)})'(\equiv \sigma(D'_{\theta,l}) \circ (\mathrm{id} \otimes \Theta^{(l)})) = D'_{\theta,l} \circ D^{(l,0)}$. Hence we prove

Lemma 1.3. $\sigma(D'_{\theta,l}) \circ (\mathrm{id} \otimes \Theta^{(l)}) = D'_{\theta,l} \circ D_\theta^{(l,0)}$.

Proof. Consider the following diagram:

$$\begin{tikzcd}
J_1(E) \arrow[rr, "\rho_1(\Theta^{(l)})"] \arrow[dr, "\rho(D_\theta)"] & & J_1(G_{l,0}) \arrow[dr, "\rho(D'_{\theta,l})"] & \\
E \arrow[u, "j_1"] \arrow[r, "D_\theta"'] & T^* \otimes E \arrow[r, "D_\theta^{(l,0)}"] \arrow[ul, "i"] & G_{l,0} \arrow[u, "j_1"] \arrow[r, "D'_{\theta,l}"] & G'_{l,0}
\end{tikzcd}$$

We know that the outer diagram, the rectangle, and the two triangles commute. We show that the parallelogram commutes, i.e., that $\rho(D'_{\theta,l}) \circ \rho_1(\Theta^{(l)}) = D'_{\theta,l} \circ D_\theta^{(l,0)} \circ \rho(D_\theta)$. We already know that

$$D'_{\theta,l} \circ D_\theta^{(l,0)} \circ \rho(D_\theta) \circ j_1 = D'_{\theta,l} \circ D_\theta^{(l,0)} \circ D_\theta = \rho(D'_{\theta,l}) \circ \rho_1(\Theta^{(l)}) \circ j_1\ .$$

Since $D'_{\theta,l} \circ D_\theta^{(l,0)} = \Theta^{(1)}(D_\theta^{(l,0)})$ which is a bundle map, it follow from $\Theta^{(1)}(D_\theta^{(l,0)}) \circ \rho(D_\theta) \circ j_1 = \rho(D'_{\theta,l}) \circ \rho_1(\Theta^{(l)}) \circ j_1$ that

$$D'_{\theta,l} \circ D_\theta^{(l,0)} \circ \rho(D_\theta) = \Theta^{(1)}(D_\theta^{(l,0)}) \circ \rho(D_\theta) = \rho(D'_{\theta,l}) \circ \rho_1(\Theta^{(l)})\ .$$

Composing now on the right with i and in view of the fact that $\rho(D_\theta)\circ i = \mathrm{id}$, we have

$$\begin{aligned} D'_{\theta,l}\circ D_\theta^{(l,0)} &= D'_{\theta,l}\circ D_\theta^{(l,0)}\circ\rho(D_\theta)\circ i \\ &= \rho(D'_{\theta,l})\circ\rho_1(\Theta^{(l)})\circ i = \rho(D'_{\theta,l})\circ\sigma_1(\Theta^{(l)}) \\ &= \sigma(D'_{\theta,l})\circ\sigma_1(\Theta^{(l)}) = \sigma(D'_{\theta,l})\circ(\mathrm{id}\otimes\Theta^{(l)})\ . \end{aligned}$$

This establishes Lemma 1.3 and hence Theorem 1.4.

We now wish to show that essentially the $\Theta^{(l)}$ are covariant derivaties of the curvature. In fact, we shall find a canonical injection $\varepsilon\colon G_{l+1,0}\to J_1(G_{l,0})$ such that $\check{D}_\theta\Theta^{(l)} = \varepsilon\circ\Theta^{(l+1)}$.

We prove first

Lemma 1.4. *The following sequence is exact:*

$$J^1_{l+2}(E)\xrightarrow{\rho_{l+1}(D_\theta)} J^0_{l+1}(T^*\otimes E)\xrightarrow{\rho_1(D_\theta^{(l,0)})} J_1(G_{l,0}) \xrightarrow{h} \frac{J_1(G_{l,0})}{\rho_1(D_\theta^{(l,0)})(J^0_{l+1}(T^*\otimes E))}\longrightarrow 0\ ,$$

where h is the natural projection.

Proof. That $\rho_{l+1}(D_\theta)$ is defined was proved in the proof of Lemma 1.1, and exactness at all points but $J^0_{l+1}(T^*\otimes E)$ is clear. We first prove $\rho_1(D_\theta^{(l,0)})\circ\rho_{l+1}(D_\theta) = 0$. Consider $\alpha\in J^1_{l+2}(E)_{x_0}$ and choose $f\in \underline{E}_{x_0}$ such that $j_{l+2}f = \alpha$. Then

$$\begin{aligned} \rho_1(D_\theta^{(l,0)})\circ\rho_{l+1}(D_\theta)\alpha &= [\rho_1(D_\theta^{(l,0)})\circ\rho_{l+1}(D_\theta)\circ j_{l+2}f]_{x_0} \\ &= [j_1\circ D_\theta^{(l,0)}\circ D_\theta f]_{x_0} = [j_1\circ\Theta^{(l)}(f)]_{x_0} = \rho_1(\Theta^{(l)})\circ(j_1 f)_{x_0} = 0\ , \end{aligned}$$

since $(j_1 f)_{x_0} = 0$.

To prove exactness at $J^0_{l+1}(T^*\otimes E)$ consider the commutative diagram:

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ S^{l+2}T^*\otimes E & \xrightarrow{\delta} & S^{l+1}T^*\otimes T^*\otimes E & \xrightarrow{\sigma_1(D_\theta^{(l,0)})} & T^*\otimes G_{l,0} \\ \downarrow & & \downarrow & & \downarrow \\ J^1_{l+2}(E) & \xrightarrow{\rho_{l+1}(D_\theta)} & J^0_{l+1}(T^*\otimes E) & \xrightarrow{\rho_1(D_\theta^{(l,0)})} & J_1(G_{l,0}) \\ \downarrow{\scriptstyle \pi_{l+1}} & & \downarrow{\scriptstyle \pi_l} & & \downarrow{\scriptstyle \pi_0} \\ J^1_{l+1}(E) & \xrightarrow{\rho_l(D_\theta)} & J^0_l(T^*\otimes E) & \xrightarrow{\rho(D_\theta^{(l,0)})} & G_{l,0} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

The columns are clearly exact, and we have seen that the bottom row is exact. It therefore suffices to prove that the top row is exact.

First, we have the commutative diagram

$$\begin{array}{ccc} S^{l+1}T^* \otimes T^* \otimes E & \xrightarrow{\sigma_1(D_\theta^{(l,0)})} & T^* \otimes G_{l,0} \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} \\ T^* \otimes S^l T^* \otimes T^* \otimes E & \xrightarrow{\sigma(D_\theta^{(l,0)})} & T^* \otimes G_{l,0}\,. \end{array}$$

Here ε is the δ map acting as the identity on $T^* \otimes E$. $\varepsilon: T^* \otimes G_{l,0} \to T^* \otimes G_{l,0}$ is the identity.

On the other hand, the image of $\sigma(D_\theta^{(l,0)})$ on $S^l T^* \otimes T^* \otimes E$ is $S^l T^* \otimes T^* \otimes E / \delta(S^{l+1}T^* \otimes E)$ which is contained in $G_{l,0}$ and injects canonically into $S^{l-1}T^* \otimes \bigwedge^2 T^* \otimes E$. The composition of $\sigma(D_\theta^{(l,0)})$ and this injection is the δ map. Thus $\ker \sigma_1(D_\theta^{(l,0)}) = \ker \sigma(D_\theta^{(l,0)}) \circ \varepsilon = \ker \delta \circ \varepsilon$. But the diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ S^{l+1}T^* \otimes T^* \otimes E & \xrightarrow{\delta} & S^l T^* \otimes \bigwedge^2 T^* \otimes E \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} \\ T^* \otimes S^l T^* \otimes T^* \otimes E & \xrightarrow{\delta} & T^* \otimes S^{l-1}T^* \otimes \bigwedge^2 T^* \otimes E \end{array}$$

commutes and the last column is exact. Therefore $\ker \sigma_1(D_\theta^{(l,0)}) = \ker \delta \circ \varepsilon = \ker \varepsilon \circ \delta = \ker \delta$. Hence exactness of the top row reduces to exactness of the δ complex.

Corollary. *Let k be the isomorphism*

$$k: G_{l+1,0} \longrightarrow J_{l+1}^0(T^* \otimes E)/\rho_{l+1}(D_\theta)(J_{l+2}^1(E))$$

of Lemma 1.1. *Then the sequence*

$$0 \longrightarrow G_{l+1,0} \xrightarrow{\varepsilon} J_1(G_{l,0}) \xrightarrow{h} J_1(G_{l,0})/\rho_1(D_\theta^{(l,0)})(J_{l+1}^0(T^* \otimes E)) \longrightarrow 0$$

is exact, where $\varepsilon = \rho_1(D_\theta^{(l,0)}) \circ k$, and $\rho_1(D_\theta^{(l,0)})$ is induced on the quotient.

We now state

Theoerm 1.5. *We have $\check{D}_\theta \Theta^{(l)} = \varepsilon \circ \Theta^{(l+1)}$.*

Proof. We start with the defining formula $\check{D}_\theta \Theta^{(l)} = j_1 \circ \Theta^{(l)} - i \circ (\mathrm{id} \otimes \Theta^{(l)}) \circ D_\theta$. Since both sides of the equation to be proved are bundle maps, their values on a local section f at a point x_0 depend only on $f(x_0)$. Let, therefore, $f \in \underline{E}_{x_0}$ and choose $f' \in \underline{E}_{x_0}$ such that $f'(x_0) = 0$ and $(D_\theta f)(x_0) = (D_\theta f')(x_0)$.

That we can do this is guaranteed by the surjectivity of $\sigma(D_\theta)$.

We have first

$$\begin{aligned}\check{D}_\theta\Theta^{(l)}(f)(x_0) &= \check{D}_\theta\Theta^{(l)}(f-f')(x_0)\\ &= j_1\circ\Theta^{(l)}(f-f')(x_0) - i\circ(\mathrm{id}\otimes\Theta^{(l)})\circ D_\theta(f-f')(x_0)\\ &= j_1\circ\Theta^{(l)}(f-f')(x_0)\ .\end{aligned}$$

Also $\varepsilon\circ\Theta^{(l+1)}(f)(x_0) = \varepsilon\circ\Theta^{(l+1)}(f-f')(x_0)$. Since $D_\theta(f-f')(x_0)=0$, $j_{l+1}\circ D_\theta(f-f')(x_0)\in J^0_{l+1}(T^*\otimes E)_{x_0}$. Using this fact we have

$$\begin{aligned}\varepsilon\circ\Theta^{(l+1)}(f-f')(x_0) &= \rho_1(D_\theta^{(l,0)})\circ k\circ\Theta^{(l+1)}(f-f')(x_0)\\ &= \rho_{l+1}(D_\theta^{(l,0)})\circ j_{l+1}\circ D_\theta(f-f')(x_0)\\ &= j_1\circ\Theta^{(l)}(f-f')(x_0)\ .\end{aligned}$$

The last two steps follows from the fact that $j_{l+1}\circ D_\theta(f-f')(x_0)\in J^0_{l+1}(T^*\otimes E)_{x_0}$ and hence is a representative of $k^{-1}\circ\Theta^{(l+1)}$ from the definition of $\rho_{l+1}(D_\theta^{(l,0)})$ considered as acting on $J^0_{l+1}(T^*\otimes E)/\rho_{l+1}(D_\theta)(J^1_{l+2}(E))$ and from the commutativity of

$$\begin{tikzcd} J_{l+2}(E) \arrow[r, "\rho_{l+1}(D_\theta)"] & J_{l+1}(T^*\otimes E) \arrow[r, "\rho_1(D_\theta^{(l,0)})"] & J_1(G_l) \\ E \arrow[u, "j_{l+2}"] \arrow[r, "D_\theta"] \arrow[rr, bend right, "\Theta^{(l)}"'] & T^*\otimes E \arrow[u, "j_{l+1}"'] \arrow[r, "D_\theta^{(l,0)}"] & \dfrac{J_l(T^*\otimes E)}{\rho_l(D_\theta)(J^0_{l+1}(E))} \arrow[u, "j_1"'] \end{tikzcd} .$$

It follows that $\check{D}_\theta\Theta^{(l)}f = \varepsilon\circ\Theta^{(l+1)}f$.

Remark 1. Using the definition of $\Theta^{(l)}$ the equation in Theorem 1.5 can be written

$$j_1\circ D_\theta^{(l,0)}\circ D_\theta - i\circ(\mathrm{id}\otimes\Theta^{(l)})\circ D_\theta = \varepsilon\circ D_\theta^{(l+1,0)}\circ D_\theta\ .$$

We are therefore motivated to prove

Corollary. *We have*

$$j_1\circ D_\theta^{(l,0)} - i\circ(\mathrm{id}\otimes\Theta^{(l)}) = \varepsilon\circ D_\theta^{(l+1,0)}\ .$$

Proof. The equation is equivalent to

$$\rho_1(D_\theta^{(l,0)})\circ j_{l+1} - i\circ(\mathrm{id}\otimes\Theta^{(l)}) = \rho_1(D_\theta^{(l,0)})\circ k\circ\rho(D_\theta^{(l+1,0)})\circ j_{l+1}\ .$$

Let $\alpha\in T^*\otimes E$ with $\alpha(x_0)=0$. We prove equality for such α. We have $(\mathrm{id}\otimes\Theta^{(l)})\alpha = 0$ and $j_{l+1}\alpha\in J^0_{l+1}(T^*\otimes E)$. It therefore suffices to prove

$\rho_1(D_\theta^{(l,0)}) - \rho_1(D_\theta^{(l,0)}) \circ k \circ \rho(D_\theta^{(l+1,0)}) = 0$ on $J_{l+1}^0(T^* \otimes E)$. But this follows from the definition of $\rho_1(D_\theta^{(l,0)}) \circ k = \varepsilon$.

Therefore

$$j_1 \circ D_\theta^{(l,0)} - i \circ (\mathrm{id} \otimes \Theta^{(l)}) - \varepsilon \circ D_\theta^{(l+1,0)}$$

is a zero-order operator, and is identically zero if and only if its composition on the right by any symbol surjective differential operator vanishes. Composing with D_θ and comparing to Theorem 1.5 we obtain the corollary.

Remark 2. In the case $l = 1$ there is a canonical splitting of the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & G_{2,0} \arrow[r, "\varepsilon"] & J_1(\wedge^2 T^* \otimes E) \arrow[r, "\tilde{\omega}_0"] \arrow[l, bend left, dashed, "\rho(\hat{D}_\theta)"] & G_{1,0} \arrow[r] \arrow[l, bend left, dashed, "i"] & 0 . \end{tikzcd}$$

i is defined as the composition of the canonical maps

$$G_{1,0} \xrightarrow{\approx} \wedge^3 T^* \otimes E \longrightarrow T^* \otimes \wedge^2 T^* \otimes E \longrightarrow J_1(\wedge^2 T^* \otimes E) .$$

Composing the formula of Theorem 1.5 with $\rho(\hat{D}_\theta)$ we find

$$\begin{aligned} \Theta^{(2)} &= \rho(\hat{D}_\theta) \circ \varepsilon \circ \Theta^{(2)} = \rho(\hat{D}_\theta) \circ D_\theta \Theta \\ &= \rho(\hat{D}_\theta) \circ j_1 \circ \Theta - \rho(\hat{D}_\theta) \circ i \circ (\mathrm{id} \otimes \Theta) \circ D_\theta \\ &= \hat{D}_\theta \circ \Theta - \sigma(\hat{D}_\theta) \circ (\mathrm{id} \otimes \Theta) \circ D_\theta . \end{aligned}$$

Thus the corollary becomes

$$D_\theta^{(2,0)} = \hat{D}_\theta \circ D_\theta - \sigma(\hat{D}_\theta) \circ (\mathrm{id} \otimes \Theta) .$$

b) *Applications.* The above theorem indicates the precise sense in which the $\Theta^{(l)}$ are higher-order curvatures. It may be asked, however, if any information is contained in the $\Theta^{(l)}$ which is not already given by Θ (the classical curvature) especially since Theorem 4 implies that $\Theta \equiv 0$ implies $\Theta^{(l)} \equiv 0$ for all l. That they indeed do is settled by

Proposition 1.7. *Given a connection θ with curvature $\Theta \neq 0$, it is always possible to find a connection θ' with $\Theta' = \Theta$ but $\Theta'^{(2)} \neq \Theta^{(2)}$.*

Proof. Consider a local trivialization of E over an open set U with a local basis given by $e_1, \cdots, e_n$. Let $\{\theta_{\nu\mu}\}$ be the matrix of one-forms for θ. Choose some nowhere vanishing function g such that $g \equiv 1$ outside of some compact set $K \subset U$ and such that dg is not identically zero. Set $\theta'_{\nu\mu} = \theta_{\nu\mu} - g^{-1}dg\delta_{\nu\mu}$, and let θ' agree with θ outside U. Then it is easily verified that $\Theta' = \Theta$ but in view of Theorem 1.5 $\Theta'^{(2)} \neq \Theta^{(2)}$.

Our next theorem plays a key role in the next chapter.

Theorem 1.6. *Let D_θ have constant rank on E, and let $\bigcap_l \ker \Theta^{(l)}(D_\theta) = E'$. Then $D_{\theta|E'} : \underline{E'} \to \underline{T^* \otimes E'}$ is defined and $\Theta(D_{\theta|E'}) = 0$.*

We start with a lemma.

Lemma 1.5. *If D_θ has constant rank, there exists an $m \leq$ fibre* dim E *such that* $\ker \Theta^{(m)} = \ker \Theta^{(m+1)}$. *Furthermore for any $l \geq m$,* $\ker \Theta^{(m)} = \ker \Theta^{(l)} = \bigcap_k \ker \Theta^{(k)}$. *If m is so chosen, then R_{m+1} is formally integrable.*

Proof. Since $g_{i+1} = 0$ for all $i \geq 0$, the maps $\pi_s : R_t \to R_s$, $s < l$, are all injective. We have $\ker \Theta^{(l)} \subset \ker \Theta^{(l-1)}$ for all l. Since $\Theta^{(l)}$ has constant rank and E is finite dimensional, there is an $m \leq$ fibre dim E such that $\ker \Theta^{(m)} = \ker \Theta^{(m+1)}$. Then $\pi_0 \circ \pi_{m+1}(R_{m+2}) = \pi_0(R_{m+2}) = \ker \Theta^{(m+1)} = \ker \Theta^{(m)} = \pi_0(R_{m+1})$. Since π_0 is injective, it follows that $\pi_{m+1}(R_{m+2}) = R_{m+1}$. Since the symbol of R_{m+1} is zero, its δ-cohomology is zero and R_{m+1} is formally integrable. It follows from Theorem 1.1 that $\ker \Theta^{(m)} = \ker \Theta^{(l)} = \bigcap_k \ker \Theta^{(k)}$ for all $l \geq m$.

Proof of Theorem 1.6. Let m be chosen such that $\ker \Theta^{(m+1)} = \ker \Theta^{(m)}$. Then R_{m+1} is formally integrable and $g_{m+1} = 0$. From the discussion in § 0, it follows that there exist $f_1, \cdots, f_N$ of E such that $j_{m+1} f_\nu$ form a basis of R_{m+2}. But $\pi_0 : R_{m+1} \xrightarrow{\approx} E'$. and since $\pi_0 \circ j_{m+1} f_\nu = f_\nu$ and π_0 is an isomorphism, the f_ν are a local basis of E'. We have $D_\theta f_\nu = \rho(D_\theta) \circ j_1 f_\nu = 0$, which guarantees that the image of D_θ on E' lies in $T^* \otimes E'$. Indeed, $D_\theta(\eta_\nu f_\nu) = d\eta_\nu \otimes f_\nu$ and $\Theta(\eta_\nu f_\nu) = D_\theta^2(\eta_\nu f_\nu) = D_\theta(d\eta_\nu \otimes f_\nu) = d^2\eta_\nu \otimes f_\nu = 0$.

Corollary. *We can induce a connection*

$$D_{\theta|E/E'} : \underline{E/E'} \longrightarrow \underline{T^* \otimes (E/E')} \,.$$

Remark 1. It is not true in general that $D_{\theta|\ker\Theta} : \underline{\ker \Theta} \to \underline{T^* \otimes \ker \Theta}$ is defined. Consider, for example, the connection on the trivial bundle over an open set U in the plane defined by $D_\theta e_1 = dx \otimes e_2$, $D_\theta e_2 = y dx \otimes (e_1 + e_2)$. Then $\Theta e_1 = 0$, $\Theta e_2 = dy dx \otimes (e_1 + e_2)$, so the kernel of Θ is generated by e_1. But by definition, $D_\theta e_1 = dx \otimes e_2 \notin T^* \otimes \ker \Theta$.

Remark 2. One might hope that $\bigcap_l \ker \Theta^{(l)}(D_{\theta|E/E'}) = 0$. However, consider the connection defined by $D_\theta e_1 = 0$, $D_\theta e_2 = y dx \otimes e_1$. Then E' is generated by e_1 and under the quotient $D_{\theta|E/E'} e_2 = 0$. It follows that $\Theta(D_{\theta|E/E'}) = 0$ and hence $\bigcap_l \ker \Theta^{(l)}(D_{\theta|E/E'}) = E/E'$.

Remark 3. However in the Riemannian case E splits into $E = E' \oplus E''$ and $D_\theta : \underline{E''} \to \underline{T^* \otimes E''}$. ($E''$ is just the orthogonal complement of E'.) In fact this is part of the de Rham decomposition of $T(M)$ (see, e.g., [4, Theorem 5.4]).

Unless $l = 1$, the bundles $G_{l,0}$ depend on the connection, creating a difficulty in comparing higher curvatures of different connections. However, recall that given a connection we have an injection $\varepsilon_l : G_{l,0} \to J_1(G_{l-1,0})$ and in particular $\varepsilon_2 : G_{2,0} \to J_1(G_{1,0}) \approx J_1(\bigwedge^2 T^* \otimes E)$.

We adopt the notation $\tilde{J}_k(F) = \underbrace{J_1(J_1(\cdots J_1}_{k \text{ times}}(F)))$. Applying the functor J_1 we have $J_1(\varepsilon_2): J_1(G_{2,0}) \to J_1(J_1(\wedge^2 T^* \otimes E)) = \tilde{J}_2(\wedge^2 T^* \otimes E)$ and $\bar{\varepsilon}_3 \equiv J_1(\varepsilon_2) \circ \varepsilon_3: G_{3,0} \to \tilde{J}_2(\wedge^2 T^* \otimes E)$. Write $\bar{\varepsilon}_2 = \varepsilon_2$ and define $\bar{\varepsilon}_l$ inductively by $\bar{\varepsilon}_l = J_1(\bar{\varepsilon}_{l-1}) \circ \varepsilon_l: G_{l,0} \to \tilde{J}_{l-1}(\wedge^2 T^* \otimes E)$. Clearly $\bar{\varepsilon}_l$ in an injection. Set $\Omega^{(l)} = \bar{\varepsilon}_l \circ \Theta^{(l)}$, and notice that $\ker \Theta^{(l)} = \ker \Omega^{(l)}$.

Theorem 1.7. *Suppose* $\Omega^{(l)}: E \to \tilde{J}_{l-1}(\wedge^2 T^* \otimes E)$ *are given and* $\bigcap_l \ker \Omega^{(l)} = 0$. *Then there is at most one connection θ such that* $\Omega^{(l)} = \bar{\varepsilon}_l \circ \Theta^{(l)}(D_\theta)$.

Proof. It is enough to look at a point x_0 in the base space. Let $\ker \Omega^{(m)}_{x_0} = \ker \Omega^{(m+1)}_{x_0} = 0$. We shall prove that $\Omega^{(m)}_{x_0}$ and $\Omega^{(m+1)}_{x_0}$ determine θ_{x_0}. Recall that $\varepsilon_{m+1} \circ \Theta^{(m+1)} = \check{D}_\theta \Theta^{(m)} = j_1 \circ \Theta^{(m)} - \sigma_1(\Theta^{(m)}) \circ D_\theta$. Composing on the left side by $J_1(\bar{\varepsilon}_m)$, we obtain $\Omega^{(m+1)} = \bar{\varepsilon}_{m+1} \circ \Theta^{(m+1)} = J_1(\bar{\varepsilon}_m) \circ \varepsilon_{m+1} \circ \Theta^{(m+1)} = J_1(\bar{\varepsilon}_m) \circ j_1 \circ \Theta(m - J_1(\bar{\varepsilon}_m) \circ \sigma_1(\Theta^{(m)}) \circ D_\theta = j_1 \circ \bar{\varepsilon}_m \circ \Theta^{(m)} - \sigma_1(\bar{\varepsilon}_m \circ \Theta^{(m)}) \circ D_\theta = j_1 \circ \Omega^{(m)} - \sigma_1(\Omega^{(m)}) \circ D_\theta$. Applying both sides to a local section f of $\underline{E}_{x_0}$ gives $\Omega^{(m+1)}(f)(x_0) - j_1 \circ \Omega^{(m)}(f)(x_0) = -\sigma_1(\Omega^{(m)}) \circ D_\theta f(x_0) = -i \circ (\mathrm{id} \otimes \Omega^{(m)})(D_\theta f)(x_0)$. This determines $D_\theta f(x_0)$ since $i \circ (\mathrm{id} \otimes \Omega^{(m)})$ is injective.

Remark. If the $\Omega^{(m)}$ has constant rank, it is enough to use $\Omega^{(m)}$ and $\Omega^{(m+1)}$ where $m =$ fibre $\dim E$.

c) *Linear connections.* If Θ is a linear connection on $E = T(M)$, then one can consider the curvature Θ as a tensor of type $(1,3)$. The covariant derivative of Θ yields a tensor of type $(1,4)$ and is denoted by $\nabla\Theta$. $\nabla^k\Theta$ denotes the kth covariant derivative of Θ. In view of the preceding results it is natural to expect a close connection between $\Theta^{(l)}$ and $\nabla^{l-1}\Theta$. (The difference in superscripts results from the fact that $\Theta^{(1)} = \Theta$.) We wish to make this relationship explicit.

Recall that ∇ can be characterized in the following way. First, ∇_x, where $X \in \Gamma(T(M))$, acts on vector fields by $\nabla_X Y = i_X(D_\theta Y)$ and on functions by $\nabla_X f = Xf = i_X(df)$. ∇_X extends uniquely to a derivation on the algebra of tensor fields commuting with contraction.

Now let X_i be a basis of vector fields, X_i^* the dual basis of cotangent vectors. Define $\nabla A = X_i^* \otimes \nabla_{X_i} A$. Since the expression $\nabla_{X_i} A$ is linear over functions with respect to the variable X_i, one shows easily that ∇A is well defined. The symbol of ∇ is the identity, so ∇ is a connection in the usual sense. Furthermore, if A is a tensor valued homomorphism of $T(M)$ and $f \in \Gamma(T(M))$, then $(\nabla A)f = \nabla(A(f)) - \sigma_1(A) \circ D_\theta f$.

Our first problem is that $\Theta^{(l)}$ and $\nabla^{l-1}\Theta$ do not live in the same place. We solve this problem by replacing the $\Theta^{(l)}$ with the $\Omega^{(l)}$ defined in the previous section. We may regard the $\nabla^k\Theta$ as a section in $\mathrm{Hom}\,(T, (\bigotimes_k T^*) \otimes \wedge^2 T^* \otimes T)$, but since $\bigotimes_k T^* \otimes \wedge^2 T^* \otimes T$ injects canonically into $\tilde{J}_k(\wedge^2 T^* \otimes T)$ we will regard $\nabla^k\Theta$ as a section of $\mathrm{Hom}\,(T, \tilde{J}_k(\wedge^2 T^* \otimes T))$.

We can now state

Theorem 1.8. *We have* $\Omega^{(l+1)}|_{\ker \Omega^{(l)}} = \nabla^l \Theta|_{\ker \Omega^{(l)}}$ *and* $\ker \Omega^{(l+1)} = \bigcap_{k=0,l} \ker \nabla^k \Theta$.

We need some lemmas.

Lemma 1.6. *If F is a tensor bundle, there exists a unique section $B \in \operatorname{Hom}(F, J_1(F))$ such that for any $A \in \operatorname{Hom}(T, F)$ we have*

$$(\check{D}_\theta - \nabla)A = B \circ A .$$

Proof. Notice that

$$\begin{aligned}[(\check{D}_\theta - \nabla)A]f &= (j_1 \circ A)(f) - \sigma_1(A)(D_\theta f) - (\nabla \circ A)(f) + \sigma_1(A)(D_\theta f) \\ &= ((j_1 - \nabla) \circ A)f .\end{aligned}$$

But j_1 and ∇ are first-order operators with the same symbol, so $(j_1 - \nabla): \underline{F} \to J_1(F)$ is $\mathcal{O}$-linear, and the lemma follows.

Lemma 1.7. *Let D_θ be a connection on E, and let F, G be vector bundles. Let $A \in \operatorname{Hom}(E, F)$ and $B \in \operatorname{Hom}(F, G)$. Then*

$$\check{D}_\theta(B \circ A) = \rho_1(B) \circ \check{D}_\theta A .$$

Proof. Let f be a section of E. Then

$$\begin{aligned}(\check{D}_\theta(B \circ A))f &= j_1((B \circ A)f) - \sigma_1(B \circ A)(D_\theta f) \\ &= \rho_1(B)(j_1(Af)) - (\rho_1(B) \circ \sigma_1(A))(D_\theta f) \\ &= \rho_1(B)(j_1(Af) - \sigma_1(A)(D_\theta f)) = (\rho_1(B) \circ \check{D}_\theta A)f .\end{aligned}$$

Lemma 1.8. *If F is a tensor bundle, there exist sections $B_i^{(l)} \in \operatorname{Hom}(\tilde{J}_{l-1}(F), \tilde{J}_l(F))$, $i = 1, \cdots, l$, such that for any $A \in \operatorname{Hom}(T, F)$ we have*

$$(\check{D}_\theta^l - \nabla^l)A = B_1^{(l)} \circ D_\theta^{l-1} A + B_2^{(l)} \circ \check{D}_\theta^{l-2}(\nabla A) + \cdots + B_l^{(l)} \circ \nabla^{l-1} A .$$

Proof. We proceed by induction, and start with Lemma 1.6. Suppose the statement has been demonstrated for $k \leq l-1$. Then we have

$$\begin{aligned}&(\check{D}_\theta^l - \nabla^l)A \\ &\quad = \check{D}_\theta(\check{D}_\theta^{l-1}A) - \nabla^l A \\ &\quad = \check{D}_\theta[B_1^{(l-1)} \circ \check{D}_\theta^{l-2} A + \cdots + B_{l-1}^{(l-1)} \circ \nabla^{l-2} A + \nabla^{l-1} A] - \nabla^l A \\ &\quad = \rho_1(B_1^{(l-1)}) \circ \check{D}_\theta^{l-1} A + \cdots + \rho_1(B_{l-1}^{(l-1)}) \circ \check{D}_\theta(\nabla^{l-2} A) + (\check{D}_\theta - \nabla)(\nabla^{l-1} A) \\ &\quad = \rho_1(B_1^{(l-1)}) \circ \check{D}_\theta^{l-1} A + \cdots + \rho_1(B_{l-1}^{(l-1)}) \circ \check{D}_\theta(\nabla^{l-2} A) + B \circ \nabla^{l-1} A .\end{aligned}$$

Thus the lemma follows with $B_i^{(l)} = \rho_1(B_i^{(l-1)})$ for $i < l$ and $B_l^{(l)} = B$.

Lemma 1.9. *If F is a tensor bundle, there exist sections $B_i^{(k,l)} \in \operatorname{Hom}(\tilde{J}_i(F), \tilde{J}_l(F))$, $i = l-k, \cdots, l-1$, $0 < k \leq l$, such that for any $A \in \operatorname{Hom}(T, F)$*

$$\check{D}_\theta^k(\nabla^{l-k}A) = B_{l-k}^{(k,l)} \circ \nabla^{l-k}A + \cdots + B_{l-1}^{(k,l)} \circ \nabla^{l-1}A + \nabla^l A .$$

Proof. First by Lemma 1.6 we have

$$\check{D}_\theta(\nabla^{l-1}A) = B_{l-1}^{(1,l)} \circ \nabla^{l-1}A + \nabla^l A$$

for all l. We proceed by induction. Assume the lemma for $k \leq m$ and for all l. Then

$$\begin{aligned} \check{D}_\theta^{m+1}\nabla^{l-m-1}A &= \check{D}_\theta[B_{l-m-1}^{(m,l-1)} \circ \nabla^{l-m-1}A + \cdots + B_{l-2}^{(m,l-1)} \circ \nabla^{l-2}A + \nabla^{l-1}A] \\ &= \rho_1(B_{l-m-1}^{(m,l-1)}) \circ (\check{D}_\theta(\nabla^{l-m-1}A)) + \cdots \\ &\quad + \rho_1(B_{l-2}^{(m,l-1)}) \circ (D_\theta(\nabla^{l-2}A)) + \check{D}_\theta(\nabla^{l-1}A) \\ &= \rho_1(B_{l-m-1}^{(m,l-1)}) \circ B_{l-m-1}^{(1,l-m)} \circ \nabla^{l-m-1}A + \rho_1(B_{l-m-1}^{(m,l-1)}) \circ \nabla^{l-m}A \\ &\quad + \cdots + \rho_1(B_{l-2}^{(m,l-1)}) \circ B_{l-2}^{(1,l-1)} \circ \nabla^{l-2}A \\ &\quad + \rho_1(B_{l-2}^{(m,l-1)}) \circ \nabla^{l-1}A + B_{l-1}^{(1,l)} \circ \nabla^{l-1}A + \nabla^l A . \end{aligned}$$

This establishes the lemma with

$$\begin{aligned} B_{l-m-1}^{(m+1,l)} &= \rho_1(B_{l-m-1}^{(1,l-m)}) \circ B_{l-m-1}^{(1,l-m)} , \\ B_i^{(m+l,l)} &= \rho_1(B_{i-1}^{(m,l-1)}) + \rho_1(B_i^{(m,l-1)}) \circ B_i^{(1,i+1)} , \\ &\qquad\qquad \text{for } l-m-1 < i < l-1, \\ B_{l-1}^{(m+1,l)} &= \rho_1(B_{l-2}^{(m,l-1)}) + B_{l-1}^{(1,l)} . \end{aligned}$$

Theorem 1.9. *If F is a tensor bundle, then*

$$\ker \check{D}_\theta^l A = \bigcap_{k=0,l} \ker \nabla^k A$$

for all l; and if $f \in \ker D_\theta^{l-1}A$, then

$$(\check{D}_\theta^l A)f = (\nabla^l A)f .$$

Proof. Notice that $\pi_1 \circ \check{D}_\theta^l A = \check{D}_\theta^{l-1}A$ so that $\ker \check{D}_\theta^l A \subset \ker \check{D}_\theta^{l-1}A$. The rest follows inductively from Lemma 1.9 with $l = k$.

Proof of Theorem 1.8. We observe that $\check{D}_\theta^l\Theta = \check{D}_\theta^l\Omega^{(1)} = \Omega^{(l+1)}$. Therefore the theorem follows from Theorem 1.9.

We also call attention to

Corollary 1 to Lemma 1.9. *We have*

$$\check{D}_\theta^l A = B_0^{(l)} \circ A + B_1^{(l)} \circ \nabla A + \cdots + B_{l-1}^{(l)}\nabla^{l-1}A + \nabla^l A ,$$

where we have set $B_i^{(l)} \equiv B_i^{(l,l)}$.

We state a final corollary to Lemma 1.9, namely,

Corollary 2 to Lemma 1.9. *There exist sections $C_i^{(l)} \in \mathrm{Hom}\,(\tilde{J}_i(F), \tilde{J}_l(F))$*

such that for any $A \in \operatorname{Hom}(T, F)$

$$\nabla^l A = C_0^{(l)} \circ A + C_1^{(l)} \circ (\check{D}_\theta A) + \cdots + C_{l-1}^{(l)} \circ (\check{D}_\theta^{l-1} A) + \check{D}_\theta^l A .$$

Proof. Follows by induction from Lemma 1.9.

It should be noticed that all of the maps $B_i^{(k,l)}$ and $C_i^{(l)}$ depend on the connection. This, of course, may give them a certain amount of interest. On the other hand, in view of this dependence, it does not follow from the above two corollaries that the $\Omega^{(l)}$ and the $\nabla^l\Theta$ contain the same information.

It is clear that $\Omega^{(l+1)} = \nabla^l\Theta$ only on $\ker \Omega^{(l)}$ because outside the kernel, $\Omega^{(l+1)}$ does not lie in the right bundle. Hence Theorem 1.8 is the best one can hope for in this respect. The definition of $\nabla\Theta$ requires that θ be a connection on the tangent bundle, and it seems unreasonable to expect a generalization of $\nabla\Theta$, which would reduce to $\nabla\Theta$ for linear connections. Hence $\Theta^{(l)}$ appears to be the natural generalization to arbitrary vector bundles.

2. Cohomology of the D_θ-complex

We now consider the equation $D_\theta f = \alpha$ where $\alpha \in \Gamma(T \otimes E)$. We will show that under appropriate compatibility conditions on α this equation can always be solved locally (in the strong sense) in the C^∞ category provided D_θ has constant rank. In fact, we obtain a reduction to the case where D_θ is flat, and using this reduction we make a preliminary study of the global problem. We obtain satisfactory answers in two cases: 1) when the base manifold of E is simply connected, 2) when $E = T(M)$ where M is a Riemannian manifold with strictly positive or strictly negative sectional curvature.

If $\Theta = 0$, the compatibility conditions are provided by $D_\theta\alpha = 0$. However, if D_θ is not formally integrable, higher order conditions are required. Hence the following proposition is of interest.

Proposition 2.1. *D_θ is formally integrable if and only if $\Theta = 0$.*

Proof. This is an immediate consequence of Theorems 1.1 and 1.2.

Remark. This is a special case of a result of Quillen [5], who used the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigwedge^q T^* \otimes g_1 & \longrightarrow & \bigwedge^q T^* \otimes R_1 & \xrightarrow{\pi} & \bigwedge^q T^* \otimes E & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle -\delta} & & \Big\downarrow{\scriptstyle D} & & & & \\
 & & \delta(\bigwedge^q T^* \otimes g_1) & \longrightarrow & \bigwedge^{q+1} T \otimes E & \xrightarrow{\rho} & C^{q+1} & \longrightarrow & 0
\end{array}$$

Here R_1 is assumed to be a first-order equation such that $\pi: R_1 \to E$ is surjective. C^{q+1} and ρ are defined by the bottom row. This determines a unique first-order operator $\underline{D}: \bigwedge^q T^* \otimes E \to C^{q+1}$ such that $\underline{D}\pi = \rho D$. Then by definition $K(R_1) = \underline{D}D: \bigwedge^q T^* \otimes R_1 \to C^{q+2}$, and $K(R_1)$ is $\bigwedge T^*$ linear. Then Quillen proved

Proposition 2.2 (*Quillen* [5, *Proposition* 15.1]). *The sequence of $\mathcal{O}$-modules*

$$R_2 \xrightarrow{\pi} R_1 \xrightarrow{K(R_1)} C^2$$

is exact.

It is easily checked that for D_θ, $\Theta \circ \pi_0 = K(R_1)$ and $C^2 = \bigwedge^2 T^* \otimes E$.

In [26] Goldschmidt proved that there are an essentially unique bundle G_l and an operator $D_\theta^{(l)}$ such that

$$E \xrightarrow{D_\theta} T^* \otimes E \xrightarrow{D_\theta^{(l)}} G_l$$

is formally exact. We quote his theorem as follows:

Theorem 2.1 (*Goldschmidt* [2b, *Theorem* 3]). *Let $\varphi: J_k(E) \to F$ be a regular differential operator of order k from E to F, and let $D_0 = \varphi \circ j_k$. Then there exists a formally exact complex*

$$(2.1) \quad 0 \longrightarrow \mathcal{S} \xrightarrow{j_k} \underline{E} \xrightarrow{D_0} \underline{G}_0 \xrightarrow{D_1} \underline{G}_1 \xrightarrow{D_2} \underline{G}_2 \xrightarrow{D_3} \cdots \underline{G}_{r-1} \xrightarrow{D_r} \underline{G}_r \cdots$$

where G_r is a vector bundle, $G_0 = F_1$, and $D_r = \Psi_r \circ j_{l_r}: \underline{G}_{r-1} \to \underline{G}_r$ is a differential operator of order l_r; moreover the sequences

$$(2.2) \quad \begin{aligned} 0 \longrightarrow R_{k+m} \longrightarrow J_{k+m}(E) \xrightarrow{\rho_m(\varphi)} J_m(G_0) \xrightarrow{\rho_{m-l}(\Psi_1)} J_{m-l_1}(G_1) \longrightarrow \\ \cdots \longrightarrow J_{m-l_1-\cdots-l_r}(G_r) \longrightarrow \cdots \end{aligned}$$

are exact at R_{k+m} and $J_{k+m}(E)$ for $m \geq 0$, at $J_m(G_0)$ for $m \geq l_1$, and at $J_{m-l_1-\cdots-l_r}(G_r)$ for $m \geq l_1 + \cdots + l_{r+1}$, $r \geq 1$.

Furthermore if the maps $\pi_m: R_{m+1} \to R_m$ have constant rank for all $m \geq k_1$, the cohomology of (2.1) *is isomorphic to the Spencer cohomology of R_k.*

The operators and bundles are constructed recursively in the following manner. For the appropriate positive integer l_r, $G_r = J_{l_r}(G_{r-1})/\rho_{l_r}(D_{r-1})(J_{l_r+l_{r-1}}(G_{r-2}))$ and $D_r = \Psi_r \circ j_{l_r}$ where Ψ_r is the natural projection. If D_{r-1} is formally integrable, and its symbol is involutive, then l_r may be chosen to be $l_r = 1$. If D_{r-1} has involutive symbol, and its lth prolonged equation R_{k+l} is formally integrable, then l_1 may be chosen to be $l_1 = l + 1$, and l_r may be chosen to be $l_r = 1$ for $r > 1$.

We now specialize Goldschmidt's theorem to the operator D_θ, and conclude

Corollary. *Suppose $\Theta^{(l)}$ has constant rank for all l, and* $\ker \Theta^{(m)} = \ker \Theta^{(m-1)}$. *Set $H_1 = G_m = J_m(T^* \otimes E)/\rho_m(D_\theta)(J_{m+1}(E))$ and $D_1 = D_\theta^{(m)} = \tilde{\omega} \circ j_m$ where $\tilde{\omega}$ is the natural projection. Define $H_2 = J_1(G_m)/\rho_1(D_\theta^{(m)})(J_{m+1}(T^* \otimes E))$ and $D_2 = \tilde{\omega} \circ j_1: \underline{G}_m \to H_2$. Define H_l and D_l inductively for $l \geq 3$ by $H_l = J_1(H_{l-1})/\rho_1(D_{l-1})(J_2(H_{l-2}))$ and $D_l = \tilde{\omega} \circ j_1: \underline{H}_{l-1} \to H_l$. Then the complex*

$$(2.3)\quad \underline{E} \xrightarrow{D_\theta} \underline{T^* \otimes E} \xrightarrow{D_\theta^{(m)}} \underline{G_m} \xrightarrow{D_2} \underline{H_2} \xrightarrow{D_3} \cdots \longrightarrow \underline{H_{r-1}} \xrightarrow{D_r} \underline{H_r} \longrightarrow \cdots$$

is formally exact and elliptic, and its cohomology is isomorphic to the cohomology of the Spencer complex of R_m.

Proof. That (2.3) is elliptic follows from its formal exactness and the ellipticity of D_θ (see [2b]). That the operators D_l are all formally integrable follows for each l by the formal exactness of the preceding stage. Since the symbol of D_θ is involutive, the symbols of the D_l are involutive (see [1b, Proposition 4.3]).

Remark. Let ω be the sheaf of local solutions to $D_\theta f = 0$. If m is chosen so that $\ker \Theta^{(m)} = \ker \Theta^{(m-1)}$, then we recall that the Spencer complex associated to D_θ is

$$(2.4)\quad \begin{aligned} &0 \longrightarrow \omega \xrightarrow{j_{m+1}} \underline{R}_{m+1} \xrightarrow{D_0} T^* \otimes R_{m+1} \xrightarrow{D_1} \cdots \\ &\qquad \xrightarrow{D_{i-1}} \textstyle\bigwedge^i T^* \otimes R_{m+1} \longrightarrow \cdots \end{aligned}$$

where $D_i = D \circ \pi_{m+1}^{-1}$, and π_m^{-1} is the inverse of the isomorphism $\pi_m : R_{m+2} \to R_{m+1}$.

Theorem 2.2. *Let ω be as above. If* $\ker \Theta^{(m)} = \ker \Theta^{(m-1)} = E'$ *is a vector bundle, then* $\omega \subset \underline{E}'$. *Furthermore the cohomology of*

$$(2.5)\quad 0 \longrightarrow \omega \longrightarrow \underline{E'} \xrightarrow{D_\theta} \underline{T^* \otimes E'} \xrightarrow{D_\theta} \underline{\textstyle\bigwedge^2 T^* \otimes E} \xrightarrow{D_\theta} \cdots$$

is isomorphic to the cohomology of (2.4) *and hence of* (2.3). *Since* (2.5) *is locally exact,* (2.3) *is locally exact.*

Proof. $\pi_0(R_m) = E'$, and therefore R_m is also the $(m-1)$th prolonged equation of $D_{\theta|E'}$. Thus the Spencer complex of $D_{\theta|E'}$ is (2.4). (2.5) is formally exact by Theorem 2.1 (Goldschmidt) and Theorem 1.3.

Since the curvature of $D_{\theta|E'}$ is zero by Theorem 1.6, $D_{\theta|\wedge T^*\otimes E'}$ reduces to d (by introducing flat frames) so that (2.5) and (2.3) are locally exact.

It is of some interest to make the isomorphism at $T^* \otimes E$ explicit. Hence we state

Theorem 2.3. *Let $\Theta^{(l)}$ be of constant rank for $l \geq 1$, and let m be chosen such that* $\ker \Theta^{(m)} = \ker \Theta^{(m-1)}$. *Suppose $\alpha \in \Gamma(T^* \otimes E)$ such that $D_\theta^{(m)}\alpha = 0$. Then there is an $f \in \Gamma(E)$ such that $\alpha = D_\theta f + \beta$, where $\beta \in \Gamma(T^* \otimes E', M)$ and $D_\theta \beta = 0$. If $\beta = D_\theta g$, then $g \in \Gamma(M, E')$.*

Proof. Since $D_\theta^{(m)}\alpha = 0$, it follows from Proposition 1.5 that $D_\theta^{(m,0)}\alpha = \Theta^{(m)} f$ for some $f \in \Gamma(E, M)$. Thus

$$D_\theta^{(m,0)}(\alpha - D_\theta f) = D_\theta^{(m,0)}\alpha - \Theta^{(m)} f = 0 .$$

On the other hand, locally $\alpha = D_\theta h$ and we have

$$\Theta^{(m)}(h-f) = D_\theta^{(m,0)}(D_\theta h - D_\theta f) = D_\theta^{(m,0)}(\alpha - D_\theta f) = 0 .$$

Hence $h - f \in \ker \Theta^{(m)} = E'$ on the one hand and on the other $\beta = \alpha - D_\theta f = D_\theta(h-f) \in \Gamma(U, T^* \otimes E')$ if h is defined on U. It follows from this that $D_\theta \beta = 0$. If $\beta = D_\theta g$, then $\Theta^{(m)} g = D_\theta^{(m,0)} \circ D_\theta g = D_\theta^{(m,0)}(\alpha - D_\theta f) = 0$, so $g \in \Gamma(E')$.

A few applications are immediate.

Theorem 2.4. *If* $\bigcap_l \ker \Theta^{(l)} = 0$, (2.3) *is globally exact and* $\omega \equiv 0$.

Corollary. *If* $E = T(M)$, *and* D_θ *is a Riemannian connection with strictly positive or strictly negative sectional curvature, then* (2.3) *is globally exact.*

The interesting case is the case $\bigcap_l \ker \Theta^{(l)} \neq 0$. In this case we have

Theorem 2.5. *Let E be a vector bundle with connection over a simply connected manifold M, and let $\Theta^{(l)}$ have constant rank. If fibre* $\dim E' = k$, $(E' = \bigcap_l \ker \Theta^{(l)})$, *then* $H_j(M, \omega) = \bigoplus_k H_j(M, R)$, *where ω is the sheaf of germs of local solutions of* $D_\theta f = 0$, *and $H_j(M, R)$ is de Rham cohomology of M.*

Proof. In this case one can take global flat frames (see e.g. [4, Corollary 9.2]) and the complex (2.5) is the de Rham complex repeated k times.

For D_θ of constant rank the problem of calculating $H_j(M, \omega)$ reduces to the case where D_θ is flat. Here one could hope to be able to calculate $H_j(M, \omega)$ from $H_j(M, R)$, $\pi_i(M)$ and the holonomy of D_θ.

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